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OPTIMAL GAIT DESIGN FOR SYSTEMS WITH DRIFT ON SO(3)

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ABSTRACT

Geckos that jump, cats that fall, and satellites that are inertially controlled fundamentally locomote in the same way. These systems are bodies in free flight that actively reorientate under the influence of conservation of angular momentum. We refer to such bodies as inertial systems. *This work presents a novel control method for inertial systems with drift that combines geometric methods and computational control. In previous work, which focused on inertial systems starting from rest, a set of visual tools was developed that readily allowed one to design gaits. A key insight of this work was deriving coordinates, called minimum perturbation coordinates, which allowed the visual tools to be applied to the design of a wide range of motions. This paper draws upon the same insight to show that it is possible to approximately analyze the kinematic and dynamic contributions to net motion independently. This approach is novel because it uses geometric tools to support computational reduction in automatic gait generation on three-dimensional spaces.*

INTRODUCTION

The ability to move about an environment is one of the most fundamental and basic capabilities of biological and man-made systems. This work focuses on systems whose locomotion is governed by the conservation of angular momentum, which we refer to as inertial systems. Several inertial systems have classically been studied by the geometric control community [1–3]. Two such systems that have received considerable attention in the literature are the falling cat [2, 4] and inertially controlled satellite [1, 3, 5, 6]. Aside from the obvious physical differences, the means by which these systems move is quite similar; each manipulates its internal degrees of freedom in a coordinated fashion to effect rotational displacement.

This paper addresses the design of cyclic controllers that allow inertial systems to achieve desired net displacements. In particular, this work focuses on three-dimensional reorientation control of inertial systems in the presence of drift. This is a challenging control problem because 1.) the space of three-dimensional rotations is globally nonlinear, 2.) the underlying structure of the space is noncommutative, i.e., the order in which the rotations occur cannot be interchanged, and 3.) inertial systems are normally underactuated.

Figure 1: A JUMPING GECKO, FALLING CAT, AND INERTIALLY CONTROLLED SATELLITE

To address these difficulties, previous control approaches have turned to either computational or geometric methods. Computational approaches, such as optimal control, have shown success automatically generating controllers over a range of motions. Unfortunately, these approaches are often black box procedures whose solutions depend on the seed of the underlying optimization.

Our previous work [7] derived geometric methods for gait design of inertial systems. These methods make tools available that allow a control designer to use intuition to readily design gaits that produce desired net displacements. A key insight of this work made it possible to derive an optimal set of coordinates, called minimum perturbation coordinates [8, 9], that allow the visual gait design tools to be used effectively over a wide range of motions.

Drawing on the insight of this previous work, we apply minimum perturbation coordinates to inertial systems with drift. The result makes it possible to approximately analyze kinematic and dynamic contributions to net motion separately. This allows the design tools of previous work to be used to select good seeds for gait optimization. We show that using good seeds dramatically reduces the computation time of standard optimal control.

RELATED WORK

This work largely draws its origin from the locomotion literature developed by the geometric mechanics community. The work of Shapere and Wilczek [10], which studied locomotion in the framework of *gauge theory*, is largely viewed as the seminal work of geometric locomotion. In the context of locomotion, a choice of gauge corresponds to a choice of coordinates.

Murray and Sastry [11], and then Kelly and Murray [12], dropped the explicit language of gauge theory, instead adopting its modern equivalent. These works, as well as a majority of the geometric mechanics literature today, largely focus on the study of connections on principle fiber bundles. The geometric tools we use in this paper are derived from these studies.

Two key elements which arose from the geometric literature are the *reconstruction equation* and *local connection*. The reconstruction equation has been used to study a large number of locomotive systems with varied physical properties [1, 13, 14]. The local connection, which is a component of the reconstruction equation, relates kinematic contributions to locomotion. The structure in the local connection provides the basis for a collection of prior work on gait design and analysis. Shammas *et. al.* [15], used a combination of the *kinematic reconstruction equation* and Stokes Theorem to derive *height functions*, which serve as a tool for designing gaits resulting in specific net motions of a three-link system. Similar approaches have been adopted by Hatton and Choset [8, 9, 16], Melli *et. al.* [17], as well as Avon and Raz [18]. In this work, we make use of these and related tools to seed optimal gait search.

The method we use to determine optimal gaits is reduced optimal control. Prior work in related optimization methods adopt either an *Euler-Lagrange* or *maximum principle* based approach. In [19], the authors derive a set of reduced Euler-Lagrange equations based on necessary conditions of optimality. Solutions to these equations are optimal trajectories, and the controls which enable the system to follow these trajectories are the *optimal controls*.

There are numerous other related works [20–22] that discuss optimization in conjunction with reduction. In contrast to [19], many utilize the *maximum principle* as the basis for deriving conditions of optimality. One such work that is closely related to the work in this paper is presented in [23]. The authors in [23] explore optimal gait design for nonholonomic systems, using the *snake board* as the nominal example. This paper differs from [23] because we explicitly make use of the geometry of the problem in our gait optimization.

PROBLEM DEFINITION

We consider underactuated inertial systems for which conservation of angular momentum defines the relationship between changes of internal shape and corresponding motion in the world. At time equal to zero we assume the system is rotating, i.e., it has nonzero angular momentum. Our goal is to prescribe a gait which balances out this initial drift in $\mathrm{SO}(3)$. More specifically, given a fixed time period T, design a gait which best *cancels* the drift of the system at the discrete times $t = k \cdot T \;\; \forall \;\; k = 1, 2, \ldots$. We refer to this control strategy as *drift compensation*.

We demonstrate our control approach on the underactuated satellite example system shown in Figure 1. We assume that the momentum wheels which are controlled by the internal shape parameters, α_1 and α_2, have joint limits $[\alpha_{\min}, \alpha_{\max}] = [-\pi/2, \pi/2]$. We note that designing a satellite with joint limits would clearly not be advantageous. Theses constraints are included to maintain the generality of our approach to systems that have physical joint limits which cannot be avoided, e.g., a gecko, cat, or tailed robot.

BACKGROUND

For the sake of completeness, we review some basic material in geometric mechanics, but refer the reader to [24, 25] for an in depth presentation.

Lagrangian Reduction of Mechanical Systems

We assume that the systems considered in this paper have a configuration space Q with the global structure $Q = G \times M$, where G is equal to the Lie group $\mathrm{SO}(3)$, and M is the base, or shape, manifold.

The system's Lagrangian is assumed to be *invariant* with respect to the group action [12, 15, 26]. As a result, the *reduced Lagrangian* can be expressed at the group identity element as

$$l(\xi, r, \dot{r}) = \frac{1}{2}(\xi \;\; \dot{r})\tilde{M}\begin{pmatrix}\xi \\ \dot{r}\end{pmatrix},$$

where ξ is a body velocity.

Defining the generalized momentum to be $p = \frac{\partial l}{\partial \xi}$, it is possible to express the reconstruction equation, which relates base velocities $\dot{r}$ and generalized momenta to body velocities as

$$\xi = -A(r)\dot{r} + I^{-1}(r)p, \tag{1}$$

where $A(r)$ is the local connection and $I(r)$ is the local locked inertia tensor. The terms on the right-hand side of Eq. (1) correspond to the kinematic, $-A(r)\dot{r}$, and dynamic, $I^{-1}(r)p$, contributions to net motion, expressed in the body frame. We note that although these components are independent in Eq. (1), in general their contributions to group displacement cannot be analyzed separately without a priori knowledge of shape trajectories.

The reconstruction equation, along with the following equations fully specify the *reduced equations of motion*:

$$\dot{p} = \frac{1}{2}\dot{r}^T \sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T \sigma_{p\dot{r}}(r)\dot{r} + \frac{1}{2}p^T \sigma_{pp}(r)p \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{r}}\right) = \frac{\partial l}{\partial r} + \tau. \tag{3}$$

Two important points to make with respect to Eq. (2) and Eq. (3) are that when considering inertial systems that drift, the momentum map derived from the reduced Lagrangian will be conserved, but in the body frame will change as the body moves. This fact is analytically expressed in Eq. (2), which states that the momentum in the body frame changes as a function of the body velocity.

The second point is that the second-order dynamics of the system only appear in the equations which govern the time evolution of the shape variables r, i.e., Eq. (3). Without loss of generality, we make the common assumption that the shape variables are directly controllable and that the explicit dependence on shape dynamics can be dropped from the dynamic description of the system.

With the reduced equations of motion defined, we can now pose the reduced optimal control problem as

$$\min_{r,\dot{r},\xi} J(\cdot) \tag{4}$$

where

$$J(\cdot) = \int_{t_0}^{t_f} \ell(\xi(t), r(t), \dot{r}(t))dt$$

subject to Eq. (1) and Eq. (2).

Similar to [27] and [28], we solve the optimal control problem defined in Eq. (4), with constraints Eq. (1) and Eq. (2) by discretizing time and using a standard constrained optimization technique such as *sequential quadratic programming*. We set up the optimizations as follows: Specify the initial and final configurations as well as initial and final configuration velocities. Next, specify the system dynamics and any bounds on the configuration and configuration velocities. Finally, add any additional constraints. The output of the optimization is a set of configuration trajectories that minimize the user-defined cost while satisfying the equality and inequality constraints. The controls which allow the system to follow these optimal trajectories are also output. Note that for the examples in the Drift Compensation section below, optimizations are implemented using the SNOPT optimization package [29] for computational efficiency.

Kinematic Tools

Our previous work [7] focused on the development of visual gait design tools for reorientation control of *purely mechanical systems*, defined in [15] to be systems which satisfy

$$\begin{aligned} \xi &= -A(r)\dot{r} \\ \dot{p} &= p = 0. \end{aligned} \tag{5}$$

Equation (5) can be used to evaluate the effect shape changes have on body motion, but only when shape changes are predefined. Assuming that the shape change information is known, we can take the integral of Eq. (5) to yield the *body velocity integral* (BVI); representing the “forwards minus backwards” motion the position directions observe from the body frame. Unfortunately, the BVI alone does not necessarily provide information with respect to how shape changes, and more specifically, gaits, can be designed to effect *desired* net motion. This is the problem we are primarily interested in.

In previous work [15], we applied Stokes' theorem to the BVI integrated along the path of a gait. This allows the BVI line integral to be rewritten as an area integral of the curl of the rows of the local connection [15, 30]; the gait serves as the boundary of the area. The curl term in this case represents the first-order nonconservative contribution to net locomotion. The integrands of the area integrals over all closed regions of the shape space form *height functions* [8, 15].

The benefit of height functions is that they can be used as a visual tool that allows intuition to guide gait design. The disadvantage of height functions is that they do not capture information with respect to the order in which motion sequences occur. The work in [17, 18] addresses this issue by augmenting height functions with a first-order noncommutative term. The resulting functions are called *constraint curvature functions* (CCFs). Like the height functions, CCFs can be calculated over closed regions of the shape space to produce a visual tool that aides gait design. For a thorough discussion of CCFs on SO(3) see [7].

MINIMUM PERTURBATION COORDINATES ON SO(3)

Minimum perturbation coordinates correspond to a choice of body frames which best captures *average rotation* of a body over a gait cycle. This frame will typically not be rigidly attached to some portion of the locomoting system. An example of a non-attached body frame is one located at the center of mass of a multi-body system. Our prior work introduced a numerical technique to determine the optimal frame which minimizes motion due to executing gaits (we have observed such a frame has its origin near the center of mass). See [7] for details on how to perform this optimization.

Kinematic Systems

Starting with a purely mechanical system, one can easily see how minimum perturbation coordinates enable CCFs to be used to effectively design gaits.

In our previous work [7], we discuss that CCFs can be used to make first-order approximations of group displacement.[1] In body-fixed coordinates, this approximation does not necessarily provide useful information with respect to reconstructing motion in the inertial frame (due to higher-order noncommutative effects). In this case, velocities in the body frame have to be transformed into the inertial frame at each instant in time to obtain useful group displacement data. This transformation is both unintuitive as well as computationally expensive; in $SO(3)$ it adds nearly an order of magnitude to the number of computations needed to obtain displacement information.

Recent work [8] has identified that mapping the CCFs into the minimum perturbation frame mitigates noncommutative contributions to net motion quite well. This transformation causes the higher-order noncommutative terms in the expression for group displacement to tend to zero faster than the first-order term that appears in the CCFs. The CCFs in minimum perturbation coordinates can thus be used to provide a reasonable approximation of group displacement. This approximation makes it possible to obtain displacement data without explicitly mapping body velocities into the inertial frame, and thus offers a computational advantage.

To illustrate how CCFs (in minimum perturbation coordinates) can subsequently be used to design gaits, a CCF over a region of the satellite systems's shape space, along with an example gait, are plotted in Figure 2. The *gait design rules* originally identified in [15] can be used to analyze the effect of this example gait on net displacement; these rules formalize the relationship between CCFs and displacement. For example, a gait which encircles a region of the shape space in which the CCF is sign definite will produce net position space motion in the associated component; the amplitude will be positive or negative depending on the direction the gait travels. A gait which encircles a region where a CCF is equally positive and negative will produce zero net position motion.

Figure 2 shows the height function (top), represented in minimum perturbation coordinates, as well as the resulting displacement (bottom) for the θ_x component of the satellite example system. The gait shown (black circle) encircles a region of the CCF that is sign definite. According to [15], this gait would be expected to produce a nonzero displacement. The displacement plots confirm that this gait does indeed result in a relatively large magnitude net displacement.

Dynamic Systems

The reconstruction equation, Eq. (1), is composed of two terms that relate kinematic and dynamic contributions to group displacement. Transforming both of these components into the

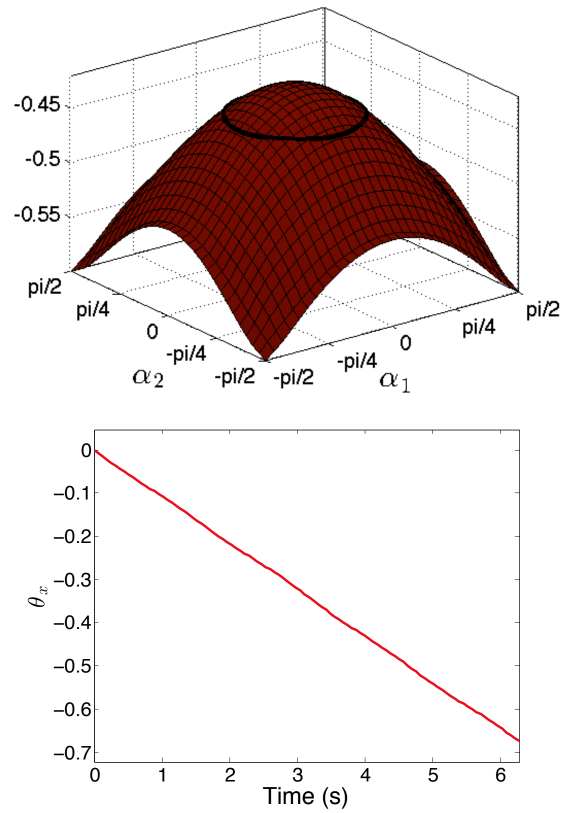

Figure 2: CCF IN MINIMUM PERTURBATION COORDINATES AND DISPLACEMENT PLOT FOR THE θ_x COMPONENT OF THE SATELLITE SYSTEM.

minimum perturbation frame, the kinematic portion remains the same as in the purely mechanical case. As we previously observed, this implies that the noncommutative contribution to group displacement arising from this term will be mitigated. Taking this one step further, we observe that mitigating kinematic noncommutative contributions over a gait cycle will in turn mitigate kinematic-dynamic cross component noncommutative contributions. It is thus reasonable to *approximately* analyze the two components of the reconstruction equation, in the minimum perturbation coordinates, as independently contributing to group displacement.

DRIFT COMPENSATION

The drift compensation control strategy is a two-step process. The first designs gaits using geometric tools and the second takes the designed gaits as seeds to an optimal control algorithm.

As noted above, the minimum perturbation coordinates make it possible to approximately treat the two components of the reconstruction equation as independently contributing to group displacement. This assumption significantly simplifies the geometric component of the control design. Assuming a fixed

[1]The group displacement is

$$g = \exp(z(\phi)) \tag{6}$$

where

$$z(\phi) = \iint_{\phi} -d\mathbf{A} + [\mathbf{A}_1, \mathbf{A}_2]\, dr + \text{ higher-order terms}, \tag{7}$$

and ϕ is a gait. The CCF is the first-order approximation to the integrand in Eq. (7).

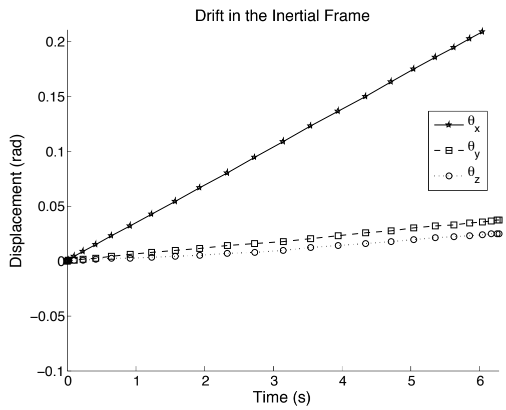

Figure 3: DISPLACEMENT DUE TO DRIFT

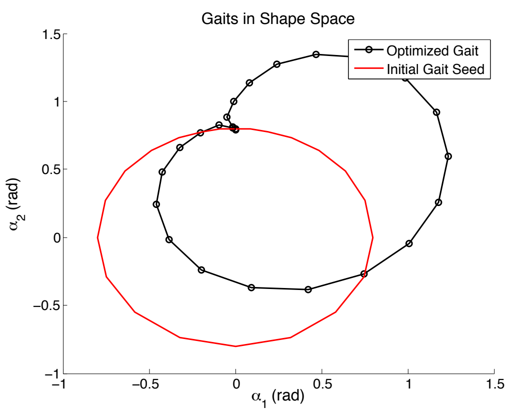

Figure 4: INITIAL SEED AND OPTIMAL GAIT DERIVED USING A BODY-FIXED FRAME IN INERTIAL COORDINATES AS DYNAMIC CONSTRAINTS OF THE GAIT OPTIMIZATION.

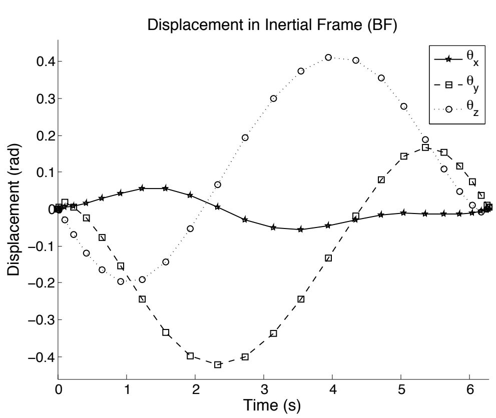

Figure 5: DISPLACEMENT OVER TIME OF A BODY-FIXED FRAME IN INERTIAL COORDINATES.

time period, the goal becomes to design a gait that approximately produces a kinematic displacement which is the inverse of the dynamic displacement due to drift, i.e., $R_{\text{gait}} = R_{\text{drift}}^{-1}$. The CCFs and gait design rules discussed in the Kinematic Systems subsection above provide the insight necessary to accomplish this.

The approximated gait solution is then used to seed the optimal control algorithm described above, which is the second step of drift compensation. The optimal control algorithm exactly solves for the gait that produces a zero-net displacement trajectory.

To illustrate the benefits of drift compensation control, we return to the satellite example system. We model this system as a cube ($l = 1$, $m = 1$) with two solid disk inertia wheels ($r = 1$, $m = 1$) located on the faces of the cube corresponding to the y- and z-body-fixed directions. We assume that the drift we wish to negate is shown in Figure 3 ($p_{\theta_x,0} = 0.035$, $p_{\theta_y,0} = 0.005$, $p_{\theta_z,0} = 0.005$). This is the drift the system would experience if not controlled. Examining the CCF in Figure 2 (as well as the CCFs corresponding to the θ_y and θ_z components shown in [7]), a circular gait, symmetric about the origin, and traveling in the clockwise direction appears to be a reasonable choice for initial gait seed.

The circular gait ($\alpha_{1,0} = 0.7$, $\alpha_{2,0} = 0$) shown in Figure 2 is redrawn in Figure 4 (solid-red line). This gait is used to seed the optimal control algorithm, where the cost function is defined to be the time integral of the norm-squared control effort. The dynamic constraints of the optimization are defined to be the dynamics of a body-fixed frame moving in the inertial coordinates, i.e.,

$$\begin{aligned} \dot{g} &= T_e L_g(-A(r)\dot{r} + I^{-1}(r)p) \\ \dot{p} &= \frac{1}{2}\dot{r}^T \sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T \sigma_{p\dot{r}}(r)\dot{r} + \frac{1}{2}p^T \sigma_{pp}(r)p. \end{aligned} \tag{8}$$

Note that in Eq. (8), the $T_e L_g$ matrix operator has to be applied to the right-hand side of Eq. (1) to obtain group velocities. The computational cost of applying this operator, which is nonlinear in the group variables, can be relatively high.

The optimal gait solution for this example is shown in Figure 4. Figure 5 shows the trajectories, in the inertial frame, that result from executing the optimized gait. We observe that the optimal gait does indeed control the system to zero net displacement at the end of the gait cycle ($t = 2\pi$).

To quantify the benefit afforded by generating good initial seeds, we compared times to completion of randomly seeded optimizations to those seeded using visual gait design tools. We ran over forty trials for each case. Our results show that on average, there is an 88% decrease in execution time when using visual tools to seed optimizations.

Additionally, we observe another potential benefit afforded by the minimum perturbation coordinates applied to this problem. In previous work [7, 8], it was observed that the BVI, when calculated in the minimum perturbation frame, provides a relatively good approximation to group displacement for kinematic

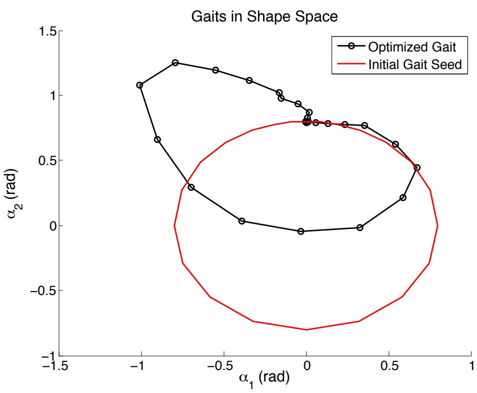

Figure 6: INITIAL SEED AND OPTIMAL GAIT DERIVED USING THE BODY VELOCITY REPRESENTED IN MINIMUM PERTURBATION COORDINATES AS THE DYNAMIC CONSTRAINT OF THE GAIT OPTIMIZATION.

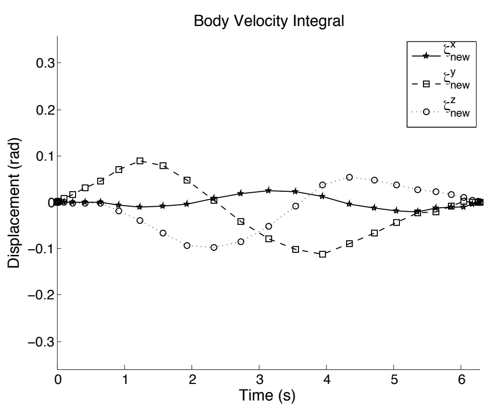

Figure 7: BODY VELOCITY INTEGRALS OVER TIME IN MINIMUM PERTURBATION COORDINATES.

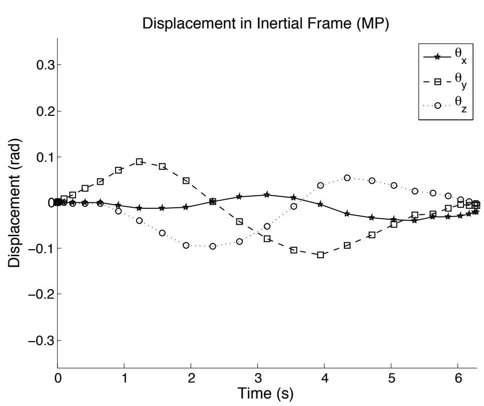

Figure 8: DISPLACEMENT OVER TIME OF THE MINIMUM PERTURBATION FRAME IN INERTIAL COORDINATES.

systems. We apply similar reasoning to the satellite system with drift. The following example shows that it is again possible to obtain good estimates of group displacement using the BVI, but where the BVI is calculated using Eq. (1) as opposed to the kinematic reconstruction equation, i.e., Eq. (5).

In this example we perform a similar analysis to that above, but optimize directly in the minimum perturbation *body* frame. Figure 6 shows the optimal gait derived using the body velocity as a dynamic constraint for the optimal control algorithm, i.e.,

$$\begin{aligned}\xi_{\text{new}} &= -A_{new}(r)\dot{r} + I^{-1}_{\text{new}}(r)p,\\ \dot{p} &= \frac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r},\text{new}}(r)\dot{r} + p^T\sigma_{p\dot{r},\text{new}}(r)\dot{r} + \frac{1}{2}p^T\sigma_{pp,\text{new}}(r)p.\end{aligned}$$

The "new" subscript here denotes objects represented in the minimum perturbation frame. The cost function for this example was again defined to be the time integral of the norm-squared control effort. The optimization was seeded with the same gait as that shown in Figure 4.

Figure 7 shows the zero-net displacement trajectories, in the minimum perturbation frame, that result from executing the optimal gait shown in Figure 6. Figure 8 shows the trajectories in Figure 7 mapped into the inertial frame. The results in Figures 7 and 8 show there is a close correlation between body frame and inertial frame trajectories. The similarity of these trajectories potentially implies that there are benefits to executing optimal control in the minimum perturbation body frame.

The first of these potential benefits is that it is theoretically possible to further reduce the computational complexity of solving optimal control problems for drift compensation. As noted above, optimizing in the inertial frame requires velocities to be transformed from the body frame via the T_eL_g operator. This transformation adds nearly an order of magnitude to the number of calculations needed to obtain displacement data. These results appear to support the hypothesis that if we are willing to sacrifice a reasonable amount of accuracy, we can perhaps dramatically reduce the computational burden of optimal control.

In addition, comparing Figure 5 to Figure 8, we observe that the intermittent displacement of the minimum perturbation frame is much smaller than that of the body-fixed frame over the optimal gait cycles. This implies that the minimum perturbation frame is a better choice for motion planning. The minimum perturbation coordinates make it possible to obtain simplified planning solutions which primarily move the system positively along the global plan.

CONCLUSION AND FUTURE WORK

The main contribution of the work presented in this paper was the development of a novel control design method for inertial reorientation systems with nonzero drift. The control approach builds upon previous work which derived geometric gait design tools for purely mechanical systems. A key insight of this prior work was the development of a set of minimum perturbation coordinates. Applying minimum perturbation coordinates to inertial systems that drift, we observed that it is possible to approximately analyze kinematic and dynamic contributions to group displacement independently. This observation makes it possible to use the gait design tools of previous work to generate initial seeds for optimal control. We observed that by choosing good initial seeds, it is possible to dramatically decrease the execution time necessary to find optimal solutions.

Initial results also imply that it is possible to approximate desired inertial trajectories using optimal controllers derived in the minimum perturbation body coordinate frame. This capability could potentially reduce the computational complexity of future gait optimization work. We propose to pursue the implications of this observation as well as to address issues of controllability for drift compensation in our future work.

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