Proceedings of ASME 2014 Dynamic Systems and Control Conference
DSCC 2014
October 22–24, 2014, San Antonio, Texas, USA

DSCC2014-6212

DISSIPATION-INDUCED SELF-RECOVERY IN SYSTEMS ON PRINCIPAL BUNDLES

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ABSTRACT

The "self-recovery" phenomenon is a seemingly curious property of certain underactuated dissipative systems in which dissipative forces always push the system to a pre-determined equilibrium state dependent on the initial conditions. The systems for which this has been studied are Abelian, with all system velocity interactions due entirely to inertial effects. In this paper we also consider Abelian systems, but in the context of principal bundles, and introduce drag in addition to inertial interactions, allowing us to show that the same conservation that induces self-recovery now depends on the trajectories of the system inputs in addition to initial conditions. We conclude by demonstrating an example illustrating the conditions derived from our proof, along with an observation that the present analysis is insufficient for self-recovery in non-Abelian systems.

INTRODUCTION

Consider Fig. 1, in which an elliptical puck with an internal mass slides on the plane while experiencing viscous friction. The internal mass can be actuated along the semimajor axis of the puck. If it is actuated to the right for example, and the puck is at rest, then momentum conservation dictates that the puck move to the left, which induces a drag force due to friction opposing the puck's movement. Surprisingly, no matter how fast the internal mass is moved or where it stops, the drag force will always push the puck back to where it started after actuation ceases.

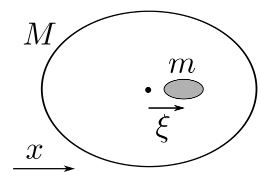

FIGURE 1: A puck sliding on a plane as a result of actuation that moves an internal mass relative to the puck.

This phenomenon, documented as *damping-induced self-recovery* by [1, 2], entails the addition of dissipation to systems exhibiting symmetries in the sense that the Lagrangian is invariant under the tangent lift of a Lie group action on the system's configuration manifold. In every case documented, the group action has been Abelian and the dissipation has been applied as a force that is linear in velocity and resists translation along the orbit of the group action. However, this dissipation was not necessarily required to exhibit the same group symmetries as those of the Lagrangian in the previous work.

In this paper we examine self-recovery in the context of a general class of mechanical systems that admit a particular geometric formalism, and we derive the conditions for self-recovery when the symmetry is Abelian. The systems we consider involve dissipation derived from a Rayleigh dissipation function that is invariant under the same lifted group action that defines the system's symmetry. Practically, this class of systems will also include those in which the local Stokes connection form, which describes system velocity interactions due to drag forces, is not necessarily zero.

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DISSIPATIVE SYSTEMS WITH SYMMETRY

The variables that specify the configuration of a mechanical system frequently admit a decomposition whereby a certain subset of variables collectively represents an element of a Lie group G. For a mobile robot, certain variables specify the robot's internal configuration while the remainder collectively specify an element of $SE(3)$—or some subgroup thereof—that encodes the robot's position and orientation relative to a laboratory frame of reference. The robot's internal configuration can be regarded as a point on a manifold M and the system's overall configuration can be regarded as a point on the configuration manifold $Q = M \times G$.

For a decomposition in this form, it can be insightful to regard Q as a principal bundle with structure group G and base M, particularly when some property of the system—typically, the Lagrangian—is invariant under translation in the group G. A system that exhibits invariance of this kind is said to exhibit a symmetry. A fundamental construction in modeling the dynamics of a system exhibiting a symmetry is that of a connection on the bundle $Q = M \times G$. This perspective is developed in detail in [3–5] and summarized below.

Consider a mobile robot with configuration manifold $Q = M \times G$ for which motion in M is directly actuated but motion in G is not, so that self-propulsion of the robot relies on the coupling of changes in its internal shape to changes in its position and orientation. In general, this coupling is dictated by a set of equations that invoke a pair of connections. A single connection is sufficient when the system's group velocity is a linear function of its shape velocity at each point in Q, provided that this function varies within Q in a specific manner as explained below. Note that a system modeled by a single connection is necessarily driftless, as motion in G requires concurrent motion in M.

The velocity with which a robot moves through its environment may be written as a vector $\dot{g} \in T_gG$ tangent to G at the robot's current position and orientation $g \in G$. If g is defined such that the robot is collocated with the laboratory frame of reference, then its position and orientation correspond to the identity element $e \in G$, and $\dot{g} \in T_eG$ may be thought of as an element of the Lie algebra $\mathfrak{g}$. Indeed, relative to a time-varying frame defined so that it always coincides with the robot's instantaneous position and orientation—a *body frame*—the robot's group velocity can always be regarded as an element of $\mathfrak{g}$.

A connection on the principal bundle $Q = M \times G$ may be specified by a Lie algebra–valued one form $A : TM \to \mathfrak{g}$ called a local connection form. The local connection form maps a tangent vector on M—a shape velocity—to a Lie algebra element. This Lie algebra element may then be equated with a group velocity $\dot{g}$ in the manner described in the preceding paragraph. Note that we assume A to be independent of g: the linear map from a shape velocity in M to a *body velocity* in $\mathfrak{g}$ depends on the configuration $q \in Q$ only through the internal configuration $r \in M$, while it is independent of the group variables.

The system in Fig. 1, for instance, admits a model based on a single connection when friction between the puck and the ground is absent. If the system begins at rest, its initial translational momentum will be zero and will remain zero regardless of actuation. If actuation moves the internal mass to the left, the puck will move to the right, stopping as soon as the actuation is discontinued. The puck's body velocity, furthermore, is proportional to the velocity of the mass relative to the puck in a manner independent of the puck's absolute position.

Since the behavior of the system in Fig. 1 is dictated by the conservation of momentum, its connection is called a *mechanical connection* and the local connection form is denoted $A_{\text{mech}} : TM \to \mathfrak{g}$. When a system is governed by a connection that represents not the conservation of momentum but a balance of viscous forces derived from the velocity gradient of a Rayleigh dissipation function—as dictates, for instance, aquatic locomotion at the extreme of low Reynolds number—the connection is instead called a *Stokes connection* and the local connection form is denoted $A_{\text{Stokes}} : TM \to \mathfrak{g}$. The latter terminology was introduced in [4].

In general, if the configuration manifold for a mechanical system admits the decomposition $Q = M \times G$, if the system's behavior is governed by Lagrange's equations subject to generalized dissipative forces derived from a Rayleigh dissipation function, and if the Lagrangian and the Rayleigh dissipation function can be written as functions of the body velocity rather than the group velocity $\dot{g}$, then the equations of motion for the system may be written in the form

$$\begin{aligned} \dot{g} &= T_eL_g \left(-A_{\text{mech}}\dot{r} + I^{-1} p\right) \\ \dot{p} &= V \left(A_{\text{Stokes}} - A_{\text{mech}}\right)\dot{r} + VI^{-1} p + \text{ad}^*_{(T_eL_g)^{-1}\dot{g}}\, p. \end{aligned} \tag{1}$$

Here the velocity $\dot{r} \in T_rM$ in the space of controlled variables represents a control input, acted upon by A_{mech} and A_{Stokes} to return Lie algebra–valued body velocities. The *local locked inertia tensor* $I : \mathfrak{g} \to \mathfrak{g}^*$, maps a body velocity into the corresponding momentum relative to the body frame. The *local viscosity tensor* $V : \mathfrak{g} \to \mathfrak{g}^*$ maps a body velocity into a viscous force and/or moment (depending on the group G) relative to the body frame. These two tensors may depend on r but not on g, reflecting the symmetries of the Lagrangian and the dissipation function. The symbol L denotes left translation in G, so that the body velocity associated with the group velocity $\dot{g}$ is given by $(T_eL_g)^{-1}\dot{g}$.

The first line in (1) defines the momentum p; the second line frames the dynamics of p in terms of the quantities described above. The symbol ad^* denotes the transpose of the Lie bracket on $\mathfrak{g}$ relative to the natural pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$. Our primary concern in the present paper is systems for which G is Abelian; in this case the ad^* vanishes from (1).

It is straightforward to verify that if dissipation were removed from a system governed by (1) ($V \to 0$), and if the system is initially at rest ($p = 0$ initially), then (1) simplifies to

$$(T_e L_g)^{-1}\dot{g} + A_{\text{mech}}\dot{r} = 0,$$

and the mechanical connection provides the linear map from the velocity $\dot{r}$ to the body velocity $(T_e L_g)^{-1}\dot{g}$. On the other hand, if the tensor $IV^{-1} : \mathfrak{g}^* \to \mathfrak{g}^*$ representing the influence of inertial effects relative to viscous effects is prepended to the second line of (1) and allowed to go to zero, (1) can be simplified to

$$(T_e L_g)^{-1}\dot{g} + A_{\text{Stokes}}\dot{r} = 0,$$

and the Stokes connection provides this linear map.

Although it is less relevant to the present paper, we also note that if dissipation is removed from the system and there is no motion in the base manifold, so that $V = 0$ and $\dot{r} = 0$, then the second line in (1) simplifies to the Lie-Poisson equation

$$\dot{p} = \text{ad}^*_{(T_e L_g)^{-1}\dot{g}}\, p$$

in rigid-body dynamics, ideal fluid mechanics, and elsewhere [6].

SELF-RECOVERY FOR ABELIAN SYSTEMS

We now establish conditions under which a system governed by (1) will exhibit dissipation-induced self-recovery. We first assume G to be Abelian, so that (1) simplifies to

$$\begin{aligned} \dot{g} &= -A_{\text{mech}}\dot{r} + I^{-1}\, p \\ \dot{p} &= V\left(A_{\text{Stokes}} - A_{\text{mech}}\right)\dot{r} + VI^{-1}\, p. \end{aligned} \tag{2}$$

This is the form of the equations governing the system in Fig. 1, for which $r = \xi$, $g = x$, and

$$\begin{aligned} A_{\text{mech}} &: \dot{\xi} \mapsto \frac{m}{M+m}\dot{\xi} \qquad & I &: \dot{x} \mapsto (M+m)\dot{x} \\ A_{\text{Stokes}} &: \dot{\xi} \mapsto 0 & V &: \dot{x} \mapsto -b\dot{x}, \end{aligned} \tag{3}$$

where b is a linear drag coefficient. Note that from the first line of (2) we can derive $p = M\dot{x} + m(\dot{x} + \dot{\xi})$, the total linear momentum.

Here we will only consider the case in which V is a constant mapping on $\mathfrak{g}$; namely, the drag force depends only on the group velocity and not the internal configuration of the system. For the system in Fig. 1, this is the 1×1 matrix containing the element $-b$. In addition, we take I to be positive definite and V to be negative definite as proper inertia and viscosity tensors, respectively. We do *not* assume that all vectors tangent to M lie in the kernel of A_{Stokes}, as is the case in (3) and in the other explicit examples appearing in the literature to date.

We first combine the two equations in (2) to eliminate p and obtain a second-order differential equation in g and r:

$$\frac{d}{dt}\left(I\dot{g} + IA_{\text{mech}}\dot{r}\right) - V\dot{g} - VA_{\text{Stokes}}\dot{r} = 0. \tag{4}$$

In the same manner as that of [1,2], this gives rise to the following conserved vector quantity (an integral of the system):

$$\mu = \underbrace{I\dot{g} + IA_{\text{mech}}\dot{r}}_{p} - Vg - V\int_{r_0}^{r(t)} A_{\text{Stokes}}\, dr. \tag{5}$$

We assume that A_{Stokes} is bounded such that its integral is defined in the time interval of interest. One can easily show that μ is conserved by taking its time derivative, which yields (4). If the system has an initial configuration (g_0, p_0), then we can evaluate the value of μ to be $\mu = p_0 - Vg_0$.

Now self-recovery only occurs after actuation is ceased, *i.e.*, $\lim_{t\to\infty}\dot{r} = 0$. After sufficient time, the connections no longer play a role in governing the system's motion. Let $\lim_{t\to\infty} r = r_f$. Applying these stipulations and rearranging (5), we obtain the simplified first-order dynamics of g:

$$\begin{aligned} \dot{g} &= I(r_f)^{-1}\left(Vg + V\int_{r_0}^{r_f} A_{\text{Stokes}}\, dr + \mu\right) \\ &= I(r_f)^{-1}V(g - g_0) + I(r_f)^{-1}\left(V\int_{r_0}^{r_f} A_{\text{Stokes}}\, dr + p_0\right). \end{aligned} \tag{6}$$

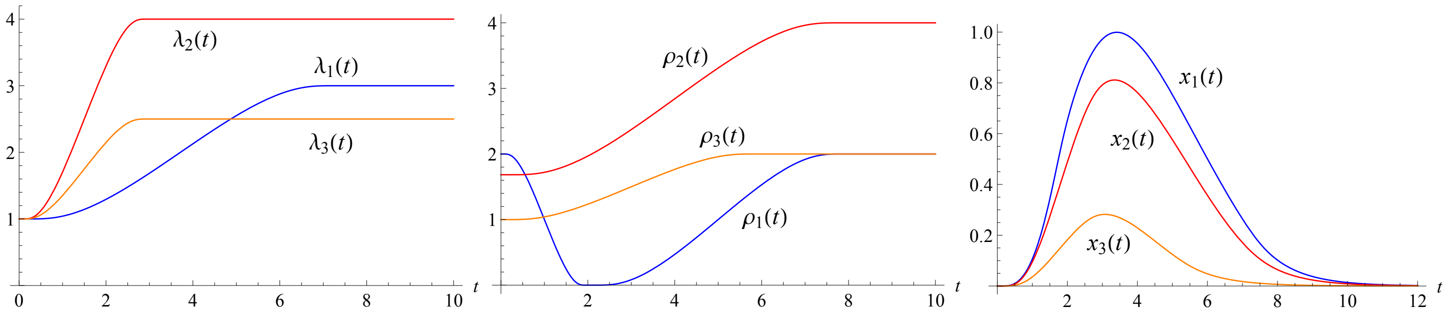
The right-hand side of (6) is an affine function in g with a unique zero, as both I and V are invertible. Due to negative definiteness of V, g will exponentially decay to this zero. In general, we can use this to explicitly find the final configuration of the system without integration of the group variables. For self-recovery, we desire $\lim_{t\to\infty} g = g_0$, which occurs when

$$V\int_{r_0}^{r_f} A_{\text{Stokes}}\, dr + p_0 = 0. \tag{7}$$

Physically, this means that the net momentum gained due to the mapping of the Stokes connection on shape inputs exactly cancels out the initial momentum p_0. Note that this formulation allows for partial recovery in some of the group variables, if only some of the rows of A_{Stokes} satisfy (7).

A SYSTEM WITH A NONZERO STOKES CONNECTION

From (7), it is clear that the system in (3) exhibits self-recovery if $p_0 = 0$. As an example of a system with nonzero



(a) Trajectories for the left leg. (b) Trajectories for the right leg. (c) Corresponding robot position trajectories.

FIGURE 3: Demonstration of damping-induced self-recovery for three different sets of shape inputs.

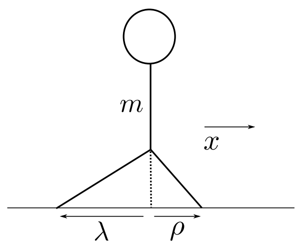


FIGURE 2: A bipedal robot that can translate laterally by actuating two internal variables λ and ρ specifying leg positions.

A_{Stokes}, consider the bipedal robot shown in Fig. 2. It is reminiscent of the inchworm in [3,4], for which V was nonzero, and the quadrupedal walker in [5], for which the system was not Abelian.

The robot's weight is supported by rigid legs, which both remain in contact with the ground but can move toward or away from the robot's body. The robot's center of mass remains at the body axis so that it can avoid falling over, and the extent to which one foot or the other provides support varies with the legs' configuration. This in turn influences the extent to which the foot sliding along the ground experiences friction.

Denote the robot's lateral position and momentum by x and p, respectively, and let the two shape variables $r = (\lambda, \rho)$, both nonnegative, describe its leg configuration as shown in Fig. 2. We model friction as a linear function of the system's velocity with proportionality constant b. The equations of motion are

$$\begin{aligned} \dot{x} &= p/m \\ \dot{p} &= -b\dot{x} - \frac{b}{\lambda+\rho}(\lambda\dot{\rho} - \rho\dot{\lambda}). \end{aligned} \tag{8}$$

From these we can extract

$$\begin{aligned} A_{\text{mech}} &: \dot{r} \mapsto 0 & I &: \dot{x} \mapsto m\dot{x} \\ A_{\text{Stokes}} &: \dot{r} \mapsto \frac{\lambda\dot{\rho} - \rho\dot{\lambda}}{\lambda+\rho} & V &: \dot{x} \mapsto -b\dot{x}. \end{aligned} \tag{9}$$

Now because A_{Stokes} is not null, self-recovery will occur only if the shape inputs satisfy (7). In the following simulations, we use $m = 1$ and $b = 1$. In addition, suppose we start from rest so that $p_0 = 0$; we thus require that $\int_{r_0}^{r_f} A_{\text{Stokes}}\, dr = 0$. Note that shape trajectories where $\lambda(t) = \rho(t)$ lie in the kernel of A_{Stokes}; physically, the drag forces on either foot cancel each other out and the system does not actually translate.

Suppose we pose the problem as follows: Given an initial configuration r_0 and final configuration r_f, find shape inputs satisfying these boundary conditions such that the robot does not have net movement from its initial position, *i.e.*, $x_f = x_0$. Along with functional form and time constraints, this can be posed as an optimal control problem. As this is beyond the scope of the present paper, we will restrict analysis to several experimentally determined smooth and monotonic functions.

Figures 3a and 3b show three sets of shape inputs. In the first scenario, we would like to increase λ_1 from 1 to 3 without effecting a net change in ρ_1. One solution would be to initially decrease ρ_1, moving the right leg to the left, followed by a more gradual restoration to its original position to nullify the gained momentum. As shown in Fig. 3c, damping pushes $x_1(t)$ back to the origin even after the legs stop moving at $t = 7$.

The second input set has both legs starting from different positions and moving outward to an equal leg placement, with λ_2 acquiring a larger net gain. With the third set of inputs, λ_3 and ρ_3 start out symmetrically, but λ_3 increases more than ρ_3. In both cases, we start actuation of both shape variables at the same time, but λ reaches its final value sooner than ρ does. Figure 3c thus shows $x_2(t)$ and $x_3(t)$ initially increasing, followed by a restoration as the right leg's actuation nullifies the gained momentum. Finally, damping pushes the system back to the origin even after we cease actuation.

SYSTEMS WITH NON-ABELIAN SYMMETRIES

We conclude by demonstrating explicitly that if the system in Fig. 1 is extended to involve a nontrivial non-Abelian symme-

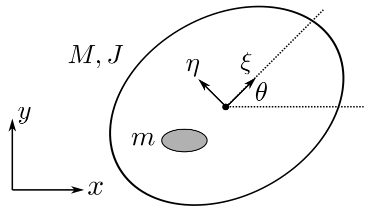


FIGURE 4: The hockey puck extended to $SE(2)$.

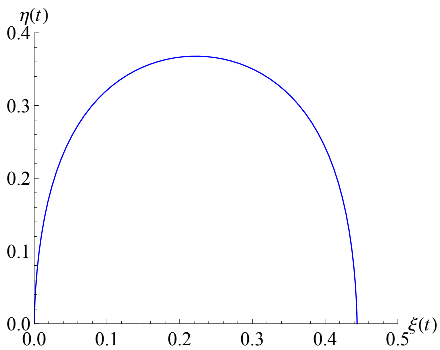


FIGURE 5: A shape trajectory for the internal mass.

try, the self-recovery phenomenon as prescribed in this paper is corrupted. Consider the modified system in Fig. 4, in which the puck, with rotational inertial J in addition to mass M, now has the full range of motion on $SE(2)$. The internal mass m can now move both longitudinally and laterally, and its position relative to the center of the puck is given by the shape variables $r = (\xi, \eta)$.

Drag forces linear in the puck's group velocities act on the puck while it is translating or rotating. It is important to note that A_{Stokes} is still the zero mapping, as is the case for the one-dimensional puck. This is because actuation of these internal shape variables does not have any effect on the Rayleigh dissipation function. If this system were Abelian, then the actual shape trajectories should not matter for self-recovery; if $p_0 = 0$, the puck should return to its starting position and orientation after actuation ceases as (7) is trivially satisfied.

For our simulation, we use the parameters $m = 1$, $M = 1$, and $J = 1$, and we assume that the drag coefficients are the same for all three group directions with $b = 1$. Starting with the system at rest, we move the internal mass as shown in Fig. 5, so that there is net displacement in ξ, the longitudinal direction, but not in η, the lateral. The resultant motion of the puck is shown in Fig. 6. Here we see that the system does not return to its original horizontal position and orientation. Even though the Stokes connection is 0, the system can build up additional momentum by virtue of the fact that its degrees of freedom do not commute, hence preventing self-recovery.

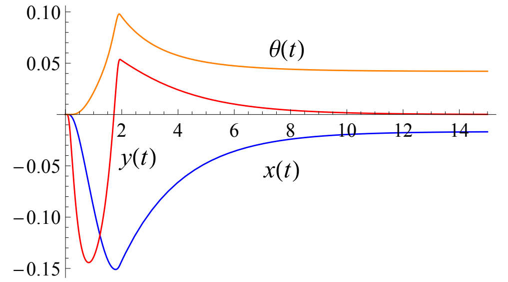


FIGURE 6: The resultant group variable trajectories.

FUTURE WORK

The damping-induced self-recovery phenomenon is, at its core, a consequence of momentum conservation in the presence of dissipation. We have extended this observation to systems on principal bundles and showed the dependence on shape trajectories when the Stokes connection is nontrivial. Further analysis may be done for systems for which V depends explicitly on the shape variables, such as a three-link swimmer experiencing both viscous drag and inertial effects. This may also be useful for designing control inputs to steer a system to a particular configuration based on (6), such as coordination of multi-agent systems.

ACKNOWLEDGMENT

Support for this work was provided in part by the National Science Foundation grant CMMI–1000652.

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